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Categorical Structures in Physics

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I give a brief exposition of some categories related to the mathematical representation of physical systems; namely smooth spaces, defined as spaces with coherent sets of curves and functions, and Hilbert geometries, considered as specializations of state spaces to the case of entities.

1. INTRODUCTION

Category theory provides a method of encoding structure in a uniform way, thereby enabling the use of general theorems on, for example, equivalence and universal constructions. In the following I shall survey some categories that arise naturally in the mathematical description of physical systems: as such objects will be taken as primitive, with morphisms being subsequently introduced to encode an existing structure. I shall therefore employ a simple set-theoretic viewpoint of category theory, although this is more a matter of convenience than a definite position on the relative foundationality of sets and categories.

I shall not repeat the standard categorical definitions to be used in the following: for details see, for example, Adámek *et al.* (1990), Borceux (1994), and MacLane (1971). In fact the categories I shall introduce will all be simple concrete categories, that is, categories with a natural faithful functor to the category *Set* of sets and maps. The objects of such categories can be reasonably thought of as being sets with additional structure which is preserved by the morphisms. As for functors, a simple example of some relevance is provided by considering posets as categories, with Hom(a,b) being a singleton if $a < b$ and empty otherwise. In this case, a functor between posets is just an isotone map.

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In a previous work (Moore, 1996) I have given a category-theoretic presentation of state spaces and property lattices in general. In this exposition I briefly discuss two specific categories which arise when one considers the paradigm state spaces for classical and quantum entities, namely smooth spaces and Hilbert geometries.

2. SMOOTH SPACES

I follow the exposition of Frölicher and Kriegl (1988); for an introductory survey see Frölicher (1982). Let S and R be sets and $\mathcal{M}$ be a fixed set of maps from S to R. An $\mathcal{M}$-space X is a triple $(A_x, \mathcal{C}_x, \mathcal{F}_x)$, where

- A_x is a set
- $C_x \subseteq \mathrm{Hom}(S; A_x)$ is a set of curves
- $\mathcal{F}_x \subseteq \mathrm{Hom}(A_x; R)$ is a set of functions

which is such that

- $\mathcal{F}_x = \Phi\mathcal{C}_x := \{f \in \mathrm{Hom}(A_x; R) | f \circ C \in \mathcal{M} \quad \forall C \in \mathcal{C}_x\}$
- $\mathcal{C}_x = \Gamma\mathcal{F}_x := \{C \in \mathrm{Hom}(S; A_x) | f \circ C \in \mathcal{M} \quad \forall f \in \mathcal{F}_x\}$

Note that $\Phi\Gamma\Phi = \Phi$ and $\Gamma\Phi\Gamma = \Gamma$. Thus, to any $\mathcal{F}_0 \subseteq \mathrm{Hom}(A_x; R)$ we can associate the $\mathcal{M}$-structure defined by $\mathcal{C}_x = \Gamma\mathcal{F}_0$ and $\mathcal{F}_x = \Phi\mathcal{C}_x$, and to any $\mathcal{C}_0 \subseteq \mathrm{Hom}(S; A_{,x})$ we can associate the $\mathcal{M}$-structure defined by $\mathcal{F}_x = \Phi\mathcal{C}_0$ and $\mathcal{C}_x = \Gamma\mathcal{F}_x$. Let $(A_x, \mathcal{C}_x, \mathcal{F}_x)$ and $(A_Y, \mathcal{C}_Y, \mathcal{F}_Y)$ be $\mathcal{M}$-spaces and g: $A_x \to A_Y$ be a map. Then g is called an $\mathcal{M}$-map if

- $\mathcal{F}_Y \circ g \circ \mathcal{C}_x \subseteq \mathcal{M}$

The resulting category of $\mathcal{M}$-spaces has several physically relevant properties. First of all, it is complete and cocomplete, with limits and colimits being constructible from the corresponding limits in the category of sets. For example, let X_j be an indexed set of $\mathcal{M}$-spaces. Then the product $\prod_j X_j$ is given by

- $A_{\prod_j x_j} = \prod_j \mathrm{A}_{x_j}$
- $\mathcal{C}_{\prod_j x_j} = \{\mathcal{C} \in \mathrm{Hom}(S; A_{\prod_j x_j}) | \pi_j \circ C \in \mathcal{C}_{x_j} \ \forall_j\}$
- $\mathcal{F}_{\prod_j x_j} = \Phi\mathcal{C}_{\prod_j x_j}$

Further, under suitable constraints on the set $\mathcal{M}$, the category is Cartesian closed (Frölicher, 1979, 1980). Physically, this means that given two $\mathcal{M}$-spaces X_1 and X_2, one can provide the function space $\mathrm{Hom}(X_1; X_2)$ with a natural $\mathcal{M}$-structure satisfying

- $\mathrm{Hom}(X_1 \prod X_2; X_3) \simeq \mathrm{Hom}(X_1; \mathrm{Hom}(X_2; X_3))$
- $\mathrm{Hom}(X_3; X_1 \prod X_2) \simeq \mathrm{Hom}(X_3; X_1) \prod \mathrm{Hom}(X_3; X_2)$

- There exists a natural morphism $\mathrm{Hom}(X_2; X_3) \prod \mathrm{Hom}(X_1; X_2) \rightarrow \mathrm{Hom}(X_1; X_3)$.

The paradigm examples of $\mathcal{M}$-spaces are l^∞ spaces, where $\mathcal{M} = l^\infty$ ($\mathbf{N}$; $\mathbf{R}$), and smooth spaces, where $\mathcal{M} = C^\infty$ ($\mathbf{R}$; $\mathbf{R}$). Both categories are indeed Cartesian closed, although for the latter the proof is nontrivial.

The category l^∞ is closely related to the category *Born* of bornological spaces. Here a bornological space is a pair $(X, \mathcal{B})$, where X is a set and $\mathcal{B}$ is a collection of subsets of X such that

- $\{x\} \in \mathcal{B}$ for each $x \in X$
- If $B_2 \in \mathcal{B}$ and $B_1 \subseteq B_2$, then $B_1 \in \mathcal{B}$
- If $B_1, B_2 \in \mathcal{B}$, then $B_1 \cup B_2 \in \mathcal{B}$

Further, a bornological map between the bornological spaces $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ is a map f: $X_1 \rightarrow X_2$ such that

- If $B_1 \in \mathcal{B}_1$, then $f(B_1) \in \mathcal{B}_2$

For more details on bornological spaces and their applications see, for example, Hogbe–Nlend (1977). In fact, there exists a natural embedding ι: $l^\infty \rightarrow$ *Born* with left adjoint η satisfying $\eta \circ \iota = \mathrm{Id}$, and such that both functors preserve the underlying spaces and maps. Further the embedding ι commutes with the functors describing the Cartesian closedness of l^∞ and *Born.*

Finally, within the category of smooth spaces one can define a general class of dualized vector spaces in which an appropriate calculus can be developed. Here a dualized vector space is a pair (E, E') where E is a vector space and E' is a subspace of the algebraic dual of E satisfying certain conditions. The smooth maps are then defined by

$$C: \quad \mathbf{R} \rightarrow E \in \mathcal{C} \text{ if and only if } m \circ C \in \mathcal{C}_\infty(\mathbf{R}; \mathbf{R}) \text{ for each } m \in E'$$

$$f: \quad E \rightarrow F \text{ is smooth if and only if } f \circ C \in \mathcal{C}_F \text{ for each } C \in \mathcal{C}_E$$

Note that on such a space one can construct:

- A locally convex topology such that E' becomes the topological dual.
- A convex bornology such that E' becomes the bornological dual.
- A convergence structure such that E' becomes the continuous dual.

3. HILBERT GEOMETRIES

A projective geometry is a set G with a ternary relation l such that (Faure and Frölicher, 1993):

- $l(p, q, p) \quad \forall p, q \in G$.
- If $l(r, p, q)$ and $l(s, p, q)$ with $p \neq q$, then $l(r, s, p)$.
- If $l(p, q, r)$ and $l(p, s, t)$, then there exists $u \in G$ such that $l(u, q, s)$ and $l(u, r, t)$.

In our context we have that $l(p, q, r)$ if $q = r$ or $p \in \{q, r\}^{\perp\perp}$. We call p, q, and r collinear if $l(p, q, r)$. A subset E of G is called a submanifold of G if for all distinct $q, r \in E$ we have that $l(p, q, r)$ implies $p \in E$. Morphisms of projective geometries are then partially defined maps f: $G_1 \backslash E_1 \to G_2$ such that:

- $E_1 \cup f^{-1}(F)$ is a linear manifold in G_1 for each linear manifold F in G_2.

For an exposition of the basic properties of the category of projective geometries see Faure and Frölicher (1995b).

A projective geometry is called irreducible if each line contains at least three points. The first fundamental theorem states that if the irreducible projective geometry G has dimension at least three, then one can construct a division ring $\mathbf{K}$ and a vector space V over $\mathbf{K}$ such that G is isomorphic to the set of 1-dimensional subspaces of V. I follow the exposition of Faure and Frölicher (1994). Fix a hyperplane H in G and define $\underline{G}$ to be the set of morphisms g: $G \backslash E \to G$ such that $g(p) = p$ for each $p \in H \backslash E$. Note that $\underline{G}$ has an induced projective structure $\underline{l}$ defined by $\underline{l}(g_1, g_2, g_3)$ if there exist p_i such that $l(g_i(q), q, p_i)$ for all $q \in G \backslash E_i$ and $l(g_1(q), g_2(q), g_3(q))$ for all $q \in \cap\{G \backslash E_i\}$. Further, define $V = \underline{G} \backslash G$ and fix a point $0 \in V$, with corresponding constant morphism $\underline{\varepsilon}$. Finally, define $\mathbf{K} = K \cup \{\varepsilon\}$, where $\underline{K}$ is the set of morphisms λ: $\underline{G} \to \underline{G}$ such that $\lambda(p) = p$ for each $p \in G$ and $\underline{l}(\lambda(p), g, 0)$ for each $g \in \underline{G}$. One can then provide $\mathbf{K}$ with a division ring structure and prove that V is a vector space over $\mathbf{K}$. For an example in two dimensions which cannot be coordinatized by a division ring see Günaydin *et al.* (1978). Finally, for a discussion of the characteristic of $\mathbf{K}$ in terms of lattice identities see Lakser (1992).

The second fundamental theorem states that for each morphism g: $G \backslash E \to G'$ between irreducible projective geometries whose image is more than a single line one can construct a semilinear map f: $V \to V'$ such that $E = \{[\phi] \mid f(\phi) = 0\}$, and for $[\phi] \notin E$ we have $g([\phi]) = [f(\phi)]$. Recall that f: $V \to V'$ is called semilinear if there exists a homomorphism s: $\mathbf{K} \to \mathbf{K}'$ such that

$$f(\phi + \lambda\psi) = f(\phi) + s(\lambda) f(\psi)$$

for each $\phi, \psi \in V$ and $\lambda \in \mathbf{K}$. We start by choosing a morphism h: $\overline{G} \backslash E \to G'$ such that $h \circ i = i' \circ g$, where i: $G \to \overline{G}$ maps a point of G to the corresponding constant map. One then proves by computation that the restriction of h to $V = \overline{G}\backslash G$ is a semilinear map f: V $\to V'$. The field homomorphism is defined by the condition that $h \circ \lambda = s(\lambda) \circ h$.

Finally, a Hilbert geometry is a projective geometry G with a binary relation $\perp$ such that (Faure and Frölicher, 1995a):

- If $p \perp q$, then $q \perp p$.
- If $r \perp p$, $s \perp p$, and $q \in r \vee s$, then $q \perp p$.
- If $p \neq q$, there exists $r \in p \vee q$ such that $r \perp p$.
- For each $p \in G$ there exists $q \in G$ with $p \not\perp q$.

One can then prove that the map

$$g: \qquad G \to G^*: \quad p \mapsto p^{\perp} = \{q | p \perp q\}$$

is a morphism, where G^* is the dual geometry of G, whose points H are hyperplanes and where $l(H_1, H_2, H_3)$ if $H_2 = H_3$ or $H_2 \cap H_3 \subseteq H_1$. The corresponding semilinear map f: $\mathcal{H} \to \mathcal{H}^*$ then defines a form

$$\langle \cdot | \cdot \rangle: \quad \mathcal{H} \times \mathcal{H} \to \mathbf{K}: \quad (\phi, \psi) \mapsto (f(\phi))(\psi)$$

Further, one can prove that the homomorphism s: $\mathbf{K} \mapsto \mathbf{K}^{\mathrm{op}}$ is invertible and so defines an involutive antiisomorphism σ: $\mathbf{K} \to \mathbf{K}$. The map $\langle \cdot | \cdot \rangle$ is then a definite Hermitian form in the sense that

$$\langle \phi | \psi + \chi \rangle = \langle \phi | \psi \rangle + \langle \phi | \chi \rangle$$

$$\langle \psi | \lambda \psi \rangle = \lambda \langle \phi | \psi \rangle$$

$$\langle \psi | \phi \rangle = \sigma(\langle \phi | \psi \rangle)$$

$$\langle \phi | \phi \rangle = 0 \quad \text{if and only if} \quad \phi = 0$$

The linear spaces V are Hermitian spaces in the sense that for each orthogonally closed subspace W of V we have that $W \oplus W^{\perp} = V$. If the underlying field is **R**, **C**, or **H**, then one can prove that V is complete in the norm topology defined by $\langle \cdot | \cdot \rangle$ and is so a Hilbert space, and that the biorthogonal subsets of V are exactly the closed submanifolds (Piron, 1964, 1976; Amemiya and Araki, 1967). To prove this, consider a vector ϕ in the completion $\overline{V}$ of V with respect to $\langle \cdot | \cdot \rangle$. Then there exists $\psi \in V$ such that $\phi = \psi - \chi$ with $\phi \perp \chi$ in $\overline{V}$. One then chooses families $\{\phi_n\}$ and $\{\chi_m\}$ in V which converge to ϕ and χ, respectively, and which are such that $\phi_n \perp \chi_m$, $\phi_n \perp \chi$, and $\phi \perp \chi_m$ for all n and m. Set $W = \{\chi_1, \chi_2, \ldots\}^{\perp}$. By hypothesis there exist $\eta \in W$ and $\zeta \in W^{\perp}$ such that $\psi = \eta + \zeta$. But one can prove that ϕ is the projection of ψ onto the closure $\overline{W}$ of W since $\phi \in \overline{W}$ and $\chi \perp W$, and this projection is just η. Since η is in V, the space is then complete.

Note that an infinite-dimensional Hermitian space is based on one of the fields **R**, **C**, or **H** if and only if it contains an infinite orthonormal sequence

(Solèr, 1995), thereby making explicit the intuition that the essential feature that distinguishes standard and nonstandard Hermitian spaces is the ability to normalize. I follow the exposition of Prestel (1995). We define $F = \{\lambda \in \mathbf{K} \mid \sigma(\lambda) = \lambda\}$ and for $\lambda, \mu \in F$ we set $\lambda \leq \mu$ if $\mu - \lambda \in \{\langle \phi | \phi \rangle | \phi \in \mathcal{H}\}$. Then, using the infinite orthonormal sequence one can prove that $\leq$ defines a Baer order: $\lambda \leq 0$ or $0 \leq \lambda$, if $0 \leq \lambda$ and $0 \leq \mu$, then $0 \leq \lambda + \mu$, if $0 \leq$, then $0 \leq \nu\lambda\sigma(\nu)$ for all $\nu \in \mathbf{K}$, and $0 \leq 1$. One then proves that the order is Archimedean in the sense that if $0 \leq \lambda < 1/n$ for all positive integers n, then $\lambda = 0$. This implies that **K** is isomorphic to a subfield of **R**, **C**, or **H** (Holland, 1977). One then proves that each Dedekind cut in F is realized in F and so **K** is equal to **R**, **C**, or **H.** Here a Dedekind cut is a subset D of F which is bounded from above and is such that if $\lambda \in D$ and $\mu \leq \lambda$, then $\mu \in D$. For an example of a nonstandard Hermitian space see Keller (1980).

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